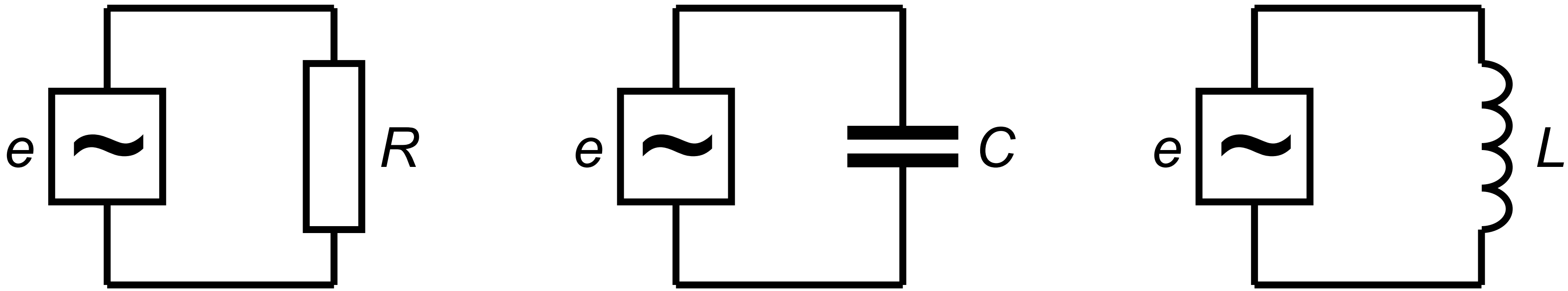
4. ELEMENTARY A.C. CIRCUITS

Main things to learn

- Resistive, inductive and capacitive circuits
- Reactance and impedance
- Phase difference between current and voltage
- Resonance



$e = \mathcal{E}_m \sin\omega t$ $\qquad$ $i = ?$

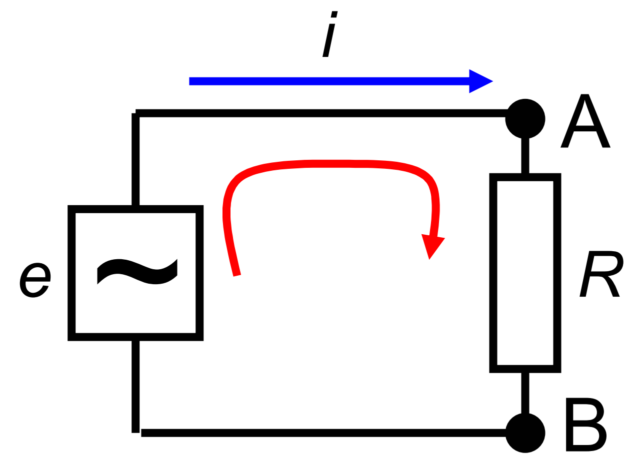
We assume that the frequency $f = \omega / 2\pi$ is small, i.e.,

voltage and current change slowly

(for frequency below 20 MHz, it is still “slowly”)

Therefore, at every moment of time Kirchhoff’s rules can be applied.

RESISTIVE CIRCUIT



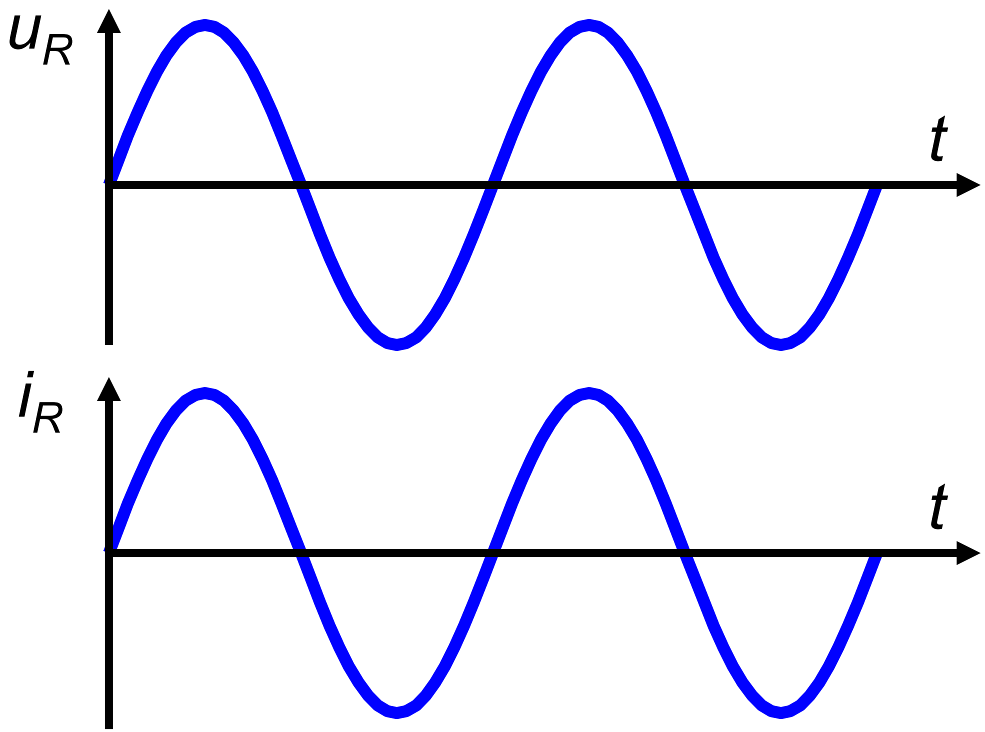
$$e = \mathcal{E}_m \sin\omega t$$

$$u_R = u_{AB} = e = U_m \sin\omega t \quad , \quad U_m = \mathcal{E}_m$$

$$iR = u_R$$

$$i = \frac{u_R}{R} = \frac{U_m}{R} \sin\omega t = I_m \sin\omega t \quad , \quad I_m = \frac{U_m}{R}$$

Resistance R acts in the same way as for direct current. Ohm's law is valid



$$u = U_m \sin\omega t$$

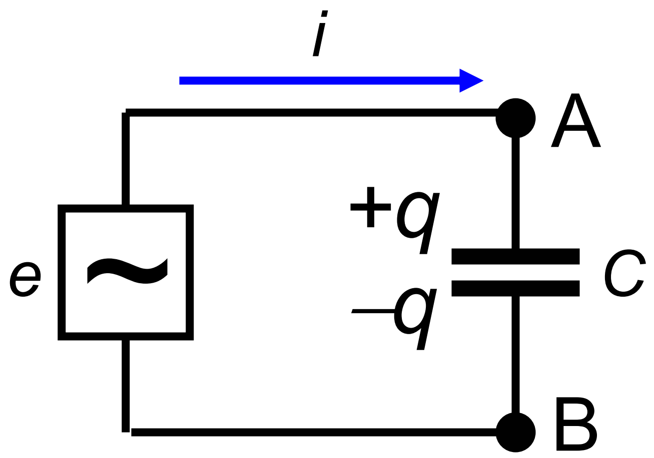
$$i = I_m \sin\omega t$$

Phase difference is zero

Current and voltage are in-phase

CAPACITIVE CIRCUIT

CAPACITIVE REACTANCE



$$e = \mathcal{E}_m \sin\omega t$$

$$u_C = u_{AB} = e = U_m \sin\omega t \quad , \quad U_m = \mathcal{E}_m$$

$$q = Cu_{AB} \quad ; \quad i = \frac{dq}{dt}$$

$$q = CU_m \sin\omega t$$

$$i = \omega CU_m \cos\omega t = I_m \sin(\omega t + \pi/2) \quad , \quad I_m = \omega CU_m$$

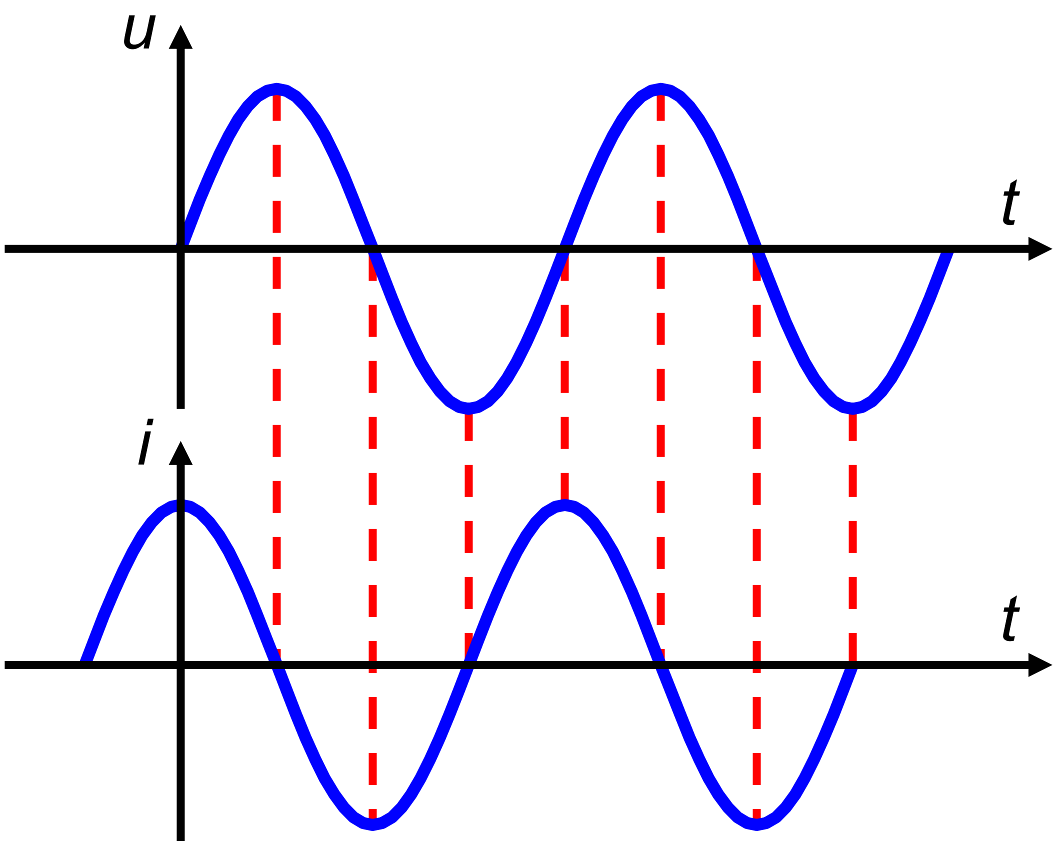
In the form similar to the Ohm's law this looks like

$I_m = \dfrac{U_m}{X_C}$, where $X_C = \dfrac{1}{\omega C}$ is equivalent to resistance

X_C is called **capacitive reactance** Units: Ohm [Ω]

X_C is proportional to $\dfrac{1}{\omega}$ X_C is proportional to $\dfrac{1}{C}$

PHASE DIFFERENCE IN A CAPACITIVE CIRCUIT



$$u = U_m \sin \omega t$$

$$i = I_m \sin(\omega t + \pi/2)$$

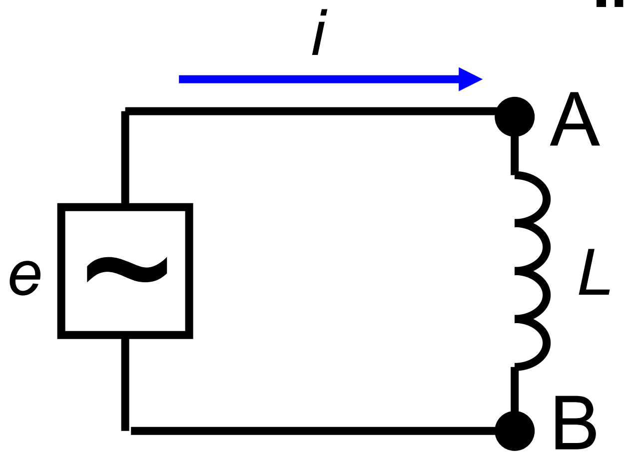
There is a **phase difference** between voltage and current

Current is ahead of voltage (current leads voltage)

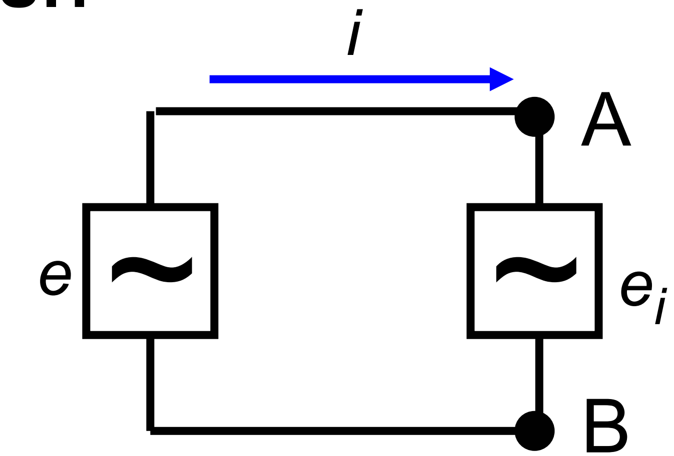
Voltage is delayed as to current (voltage lags current)

by $\pi/2$ - or 90° - or quarter-cycle

INDUCTIVE CIRCUIT



is equivalent to



$$e = \mathcal{E}_m \sin\omega t$$

$$u_{AB} = e = U_m \sin\omega t\,,\ U_m = \mathcal{E}_m$$

e_i - EMF of self-induction

$$e_i = -L\frac{di}{dt}$$ (current derivative)

2nd Kirchhoff's law

$$e + e_i = 0$$

$$U_m \sin\omega t - L\frac{di}{dt} = 0$$

INDUCTIVE REACTANCE

$$\frac{di}{dt} = \frac{U_m}{L}\sin\omega t$$

$$i = \int \frac{U_m}{L}\sin\omega t \, dt = \frac{U_m}{L}\left(-\frac{1}{\omega}\cos\omega t\right) = -\frac{U_m}{\omega L}\cos\omega t =$$

$$= \frac{U_m}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right) = I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$, where $I_m = \frac{U_m}{\omega L}$

In the form similar to the Ohm's law this looks like

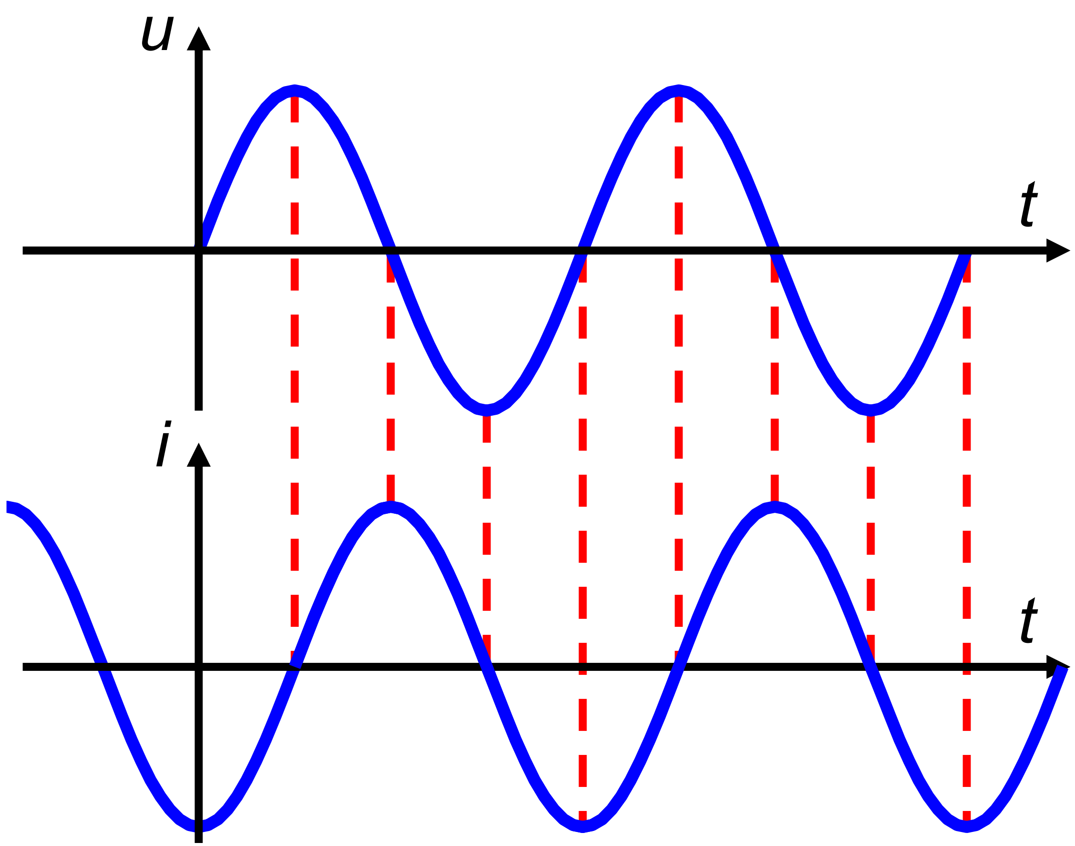
$I_m = \frac{U_m}{X_L}$, where $X_L = \omega L$ is equivalent to resistance

X_L is called **inductive reactance** Units: Ohm [Ω]

X_L is proportional to ω

X_L is proportional to L

PHASE DIFFERENCE IN AN INDUCTIVE CIRCUIT



$$u = U_m \sin \omega t$$

$$i = I_m \sin(\omega t - \pi/2)$$

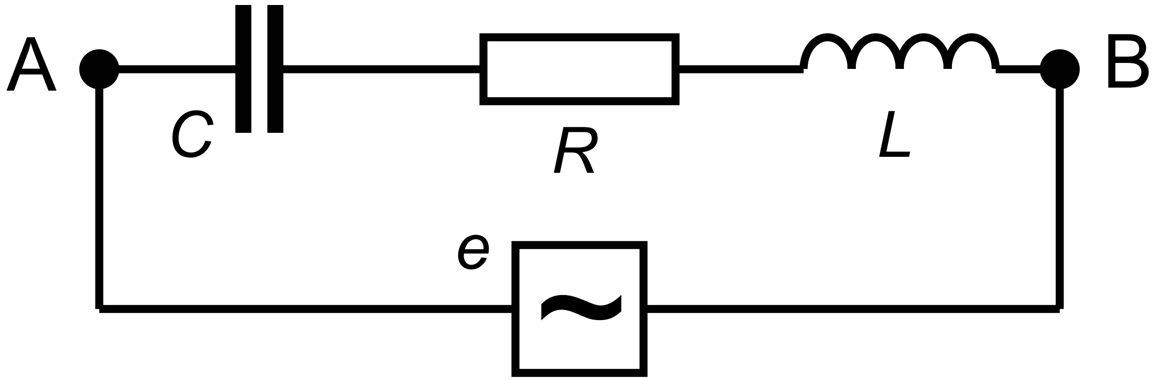
There is a **phase difference** between voltage and current

Current is behind voltage (current lags voltage)

Voltage is ahead of current (voltage leads current)

by π/2 - or 90^{o} - or quarter-cycle

EXAMPLE: EFFECT OF THE PHASE DIFFERENCE



LRC are connected in series

How to analyse this circuit?

1st Kirchhoff's law:

Current is the same $i = I_m \sin \omega t$

$$u_C = I_m X_C \sin(\omega t - \pi/2) \quad \text{where} \quad X_C = 1/\omega C$$

$$u_R = I_m R \sin \omega t$$

$$u_L = I_m X_L \sin(\omega t + \pi/2) \quad \text{where} \quad X_L = \omega L$$

2nd Kirchhoff's law:

$$e = u_{AB}$$

$$u_{AB} = u_C + u_R + u_L$$

$$u_{AB} = I_m [X_C \sin(\omega t - \pi/2) + R \sin \omega t + X_L \sin(\omega t + \pi/2)]$$

IMPEDANCE

It is possible to show that u_{AB} can be written as

$$u_{AB} = I_m Z \sin(\omega t + \varphi)$$ where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$ and $$\varphi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

$$X = \omega L - \frac{1}{\omega C}$$ - total reactance

φ - phase difference between voltage and current

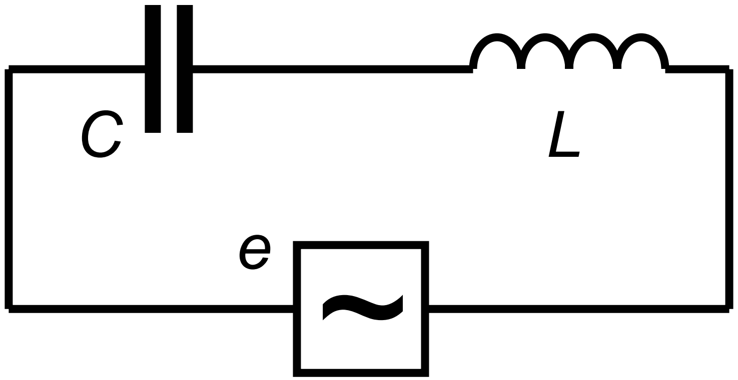
Z - equivalent to resistance - **impedance**

$$\text{Impedance} = \sqrt{\text{Resistance}^2 + \text{Reactance}^2}$$

Note that capacitive and inductive reactances are

not added but subtracted from each other

EXAMPLE: *LC* IN SERIES



$$R = 0 \quad \therefore \quad Z = X$$

$$i = I_m \sin \omega t$$

$$u = I_m \left[X_C \sin(\omega t - \pi/2) + X_L \sin(\omega t + \pi/2) \right] =$$
$$= I_m (X_L - X_C) \sin(\omega t + \pi/2)$$

- It is possible that both X_L and X_C are rather large with $Z = X = X_L - X_C$ being **much smaller**

- U_L and U_C may be much larger than $\mathcal{E}_m$

- Everything is frequency dependent: $Z = \omega L - \dfrac{1}{\omega C}$

At $\omega = \dfrac{1}{\sqrt{LC}}$ $\quad Z = 0$!!!!

At given voltage, current becomes very large - **resonance**