7. COMPLEX NOTATIONS

Main things to learn

- Complex numbers: modulus and phase
- Relation of complex numbers and phasor diagrams
- Complex impedance and admittance
- Application to problems

Imaginary unit : $j = \sqrt{-1} \quad \therefore \quad j \times j = -1$

Complex number $z = a + bj$: a - real part, b - imaginary part

Operations with complex numbers

Addition : $(a+bj)+(c+dj)=(a+c)+(b+d)j$

Subtraction : $(a+bj)-(c+dj)=(a-c)+(b-d)j$

Multiplication : $(a+bj)\times(c+dj)=(ac-bd)+(bc+ad)j$

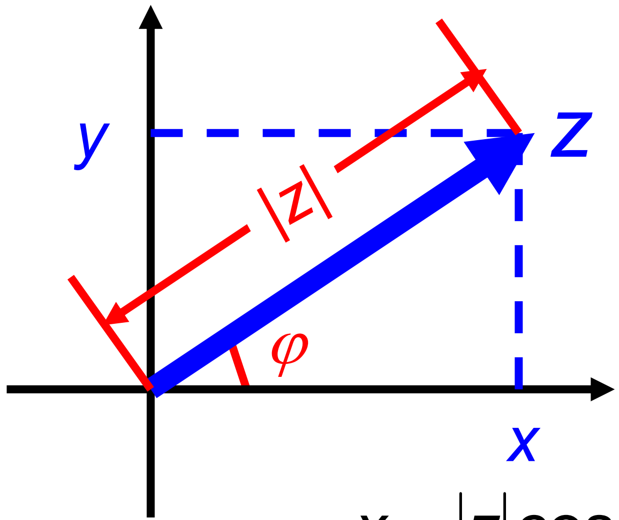
Division : $\dfrac{a+bj}{c+dj}=\dfrac{(a+bj)\times(c-dj)}{(c+dj)\times(c-dj)}=\dfrac{(ac+bd)+(bc-ad)j}{c^2+d^2}$

$c-dj$ is a **complex conjugate** to $c+dj$

MODULUS AND PHASE OF A COMPLEX NUMBER

A complex number $z = x + yj$ actually consists of two real numbers x and y

Therefore, it can be presented as a vector z in the x-y coordinate plane



The vector can be described in two ways:

1) With coordinates x and y

2) With its length $|z|$ and the angle φ

with respect to the x-axis

$$x = |z|\cos\varphi \quad ; \quad y = |z|\sin\varphi \quad ; \quad |z| = \sqrt{x^2 + y^2}$$

$$z = x + yj = |z| \times (\cos\varphi + j \times \sin\varphi) = |z| \times e^{j\varphi}$$

$|z|$ is the **modulus (amplitude)** and φ is the **phase** of the complex number z

Complex numbers can account for both amplitude and phase

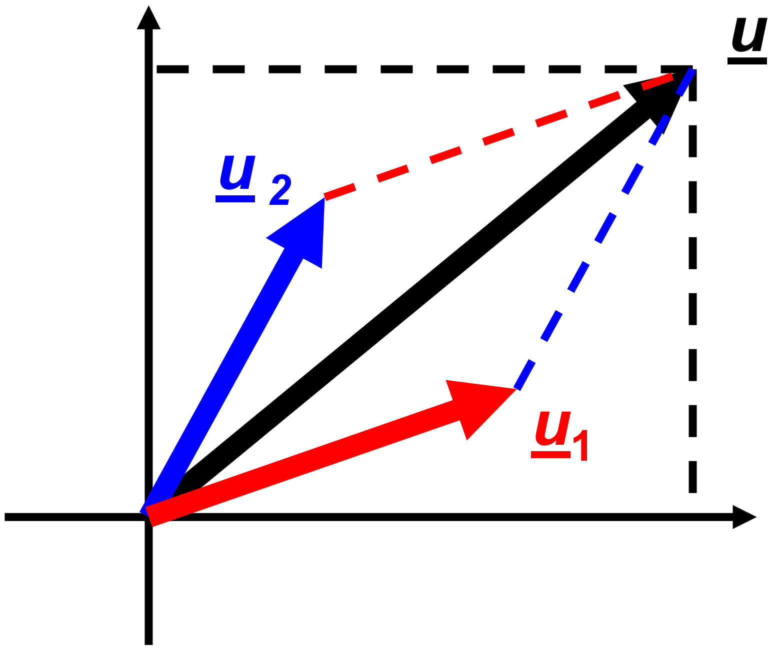
Therefore, the complex numbers can be used
to describe quantities which have both amplitude and phase
like alternating voltages, currents etc.

COMPLEX NUMBERS AND PHASOR DIAGRAMS

Complex numbers and phasors are described in a very similar way

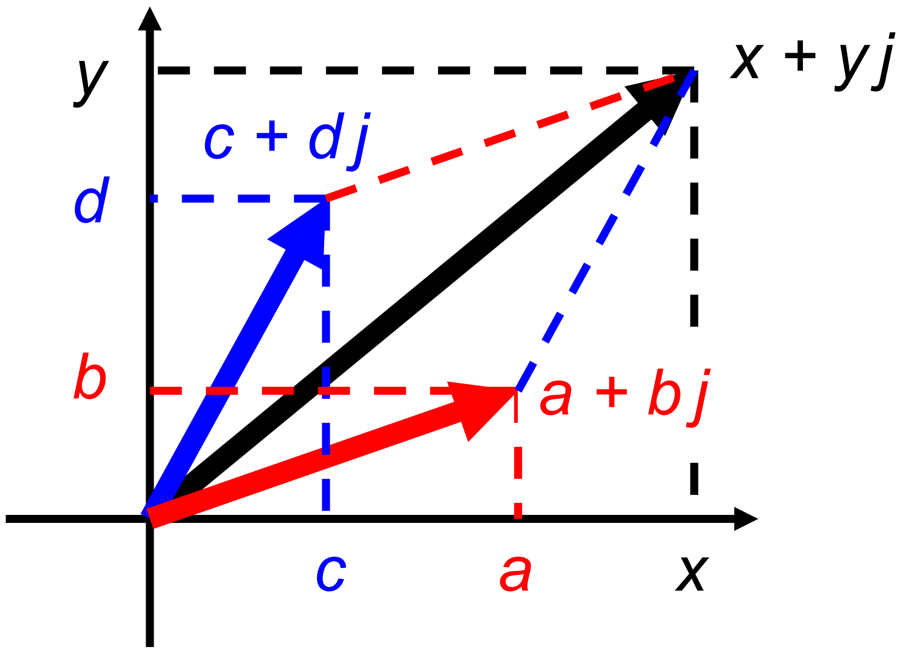
Addition of phasors:

$$\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}_1 + \underline{\boldsymbol{u}}_2$$



Addition of complex numbers

$$x + y\,j = (a + b\,j) + (c + d\,j)$$

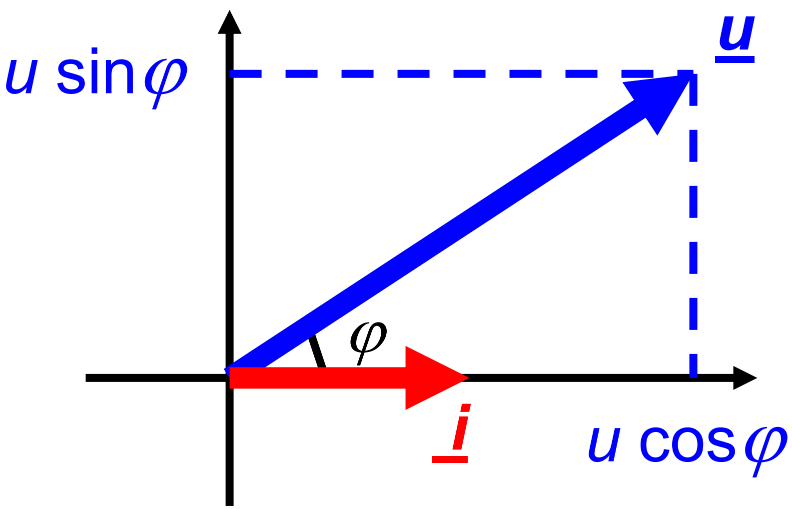


The complex numbers can be used to describe alternating quantities which have both amplitude and phase in the same way as phasors

Phasors can be represented by complex numbers

COMPLEX NOTATIONS FOR VOLTAGE

Assume that voltage leads current by the phase angle φ



Phasor-diagram representation

Current i - reference

Voltage u - at the angle φ to the x-axis

How can we represent them by complex numbers?

Current I can be represented as a real number (no y-component)

Voltage U can be represented as a complex number with the real part $U_1 = U\cos\varphi$ and the imaginary part $U_2 = U\sin\varphi$

i.e. as a complex number with the amplitude $|U|$ and the phase φ

$$U = U_1 + U_2 j = |U| \times (\cos\varphi + j \times \sin\varphi) = |U| \cdot e^{j\varphi}$$

Note that the measured voltage is always a real number.

It is represented as a complex number to take the phase into account

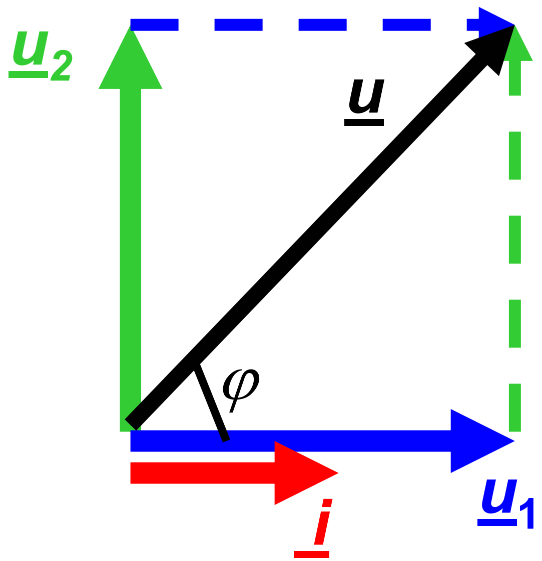
COMPLEX IMPEDANCE

Ohm's law for an a.c. circuit: $U = I\,Z$ where Z - impedance,

U - complex voltage, I - complex current

To satisfy the Ohm's law in complex notations, Z should be complex

The phase difference between voltage and current appears due to reactive elements in the circuit - capacitors and inductors



The in-phase component of voltage u_1 is due to resistance R : $U_1 = U_R = I\,R$

The quadrature component of voltage u_2 is due to reactance X : $U_2 = U_X = I\,X$

Therefore, to satisfy the Ohm's law in complex notations, we should take the impedance Z as

$$\boxed{Z = R + X\,j}$$

Resistance is the real part of impedance of the circuit

Reactance is the imaginary part of impedance of the circuit

HOW TO USE THE METHOD OF COMPLEX NOTATIONS

To each element of a circuit with resistance R and reactance X

we attribute a **complex impedance** $Z = R + X\,j$

After that, an a.c. circuit is analysed

in the same way as a d.c. circuit with the use of the complex numbers

Resitance : $X = 0 \quad \therefore \quad Z_R = R$

Inductance : $R = 0\ ,\ X_L = \omega L \quad \therefore \quad Z_L = j\omega L$

Capacitance : $R = 0\ ,\ X_C = -\dfrac{1}{\omega C} \quad \therefore \quad Z_C = -\dfrac{j}{\omega C} = \dfrac{1}{j\omega C}$

EXAMPLES

LR in series : $Z = Z_R + Z_L = R + j\omega L\,,\ |Z| = \sqrt{(\omega L)^2 + R^2}$

CR in series : $Z = Z_R + Z_C = R + \dfrac{1}{j\omega C}\,,\ |Z| = \sqrt{\left(\dfrac{1}{\omega C}\right)^2 + R^2}$

ADMITTANCE, CONDUCTANCE AND SUSCEPTANCE

Admittance Y - equivalent of conductance for a.c. circuits: $Y = \dfrac{1}{Z}$

Conductance G - active component of admittance

Susceptance B - reactive component of admittance

IN COMPLEX NOTATIONS

$$\boxed{Y = G + Bj}$$

Conductance is the real part of admittance

Susceptance is the imaginary part of admittance

$$Z = R + Xj \quad \therefore \quad Y = \frac{1}{Z} = \frac{1}{R + Xj} = \frac{R - Xj}{R^2 + X^2} = \frac{R - Xj}{\left|Z^2\right|}$$

$$G = \frac{R}{\left|Z^2\right|} \quad \text{and} \quad B = -\frac{X}{\left|Z^2\right|}$$

For parallel circuits (elements 1 and 2 in parallel)

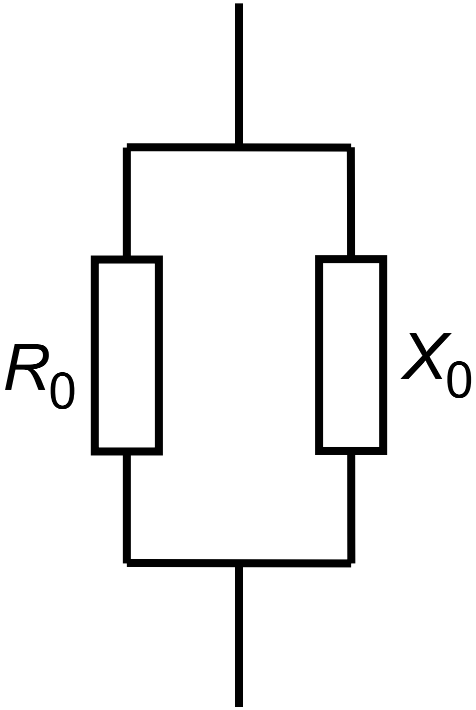
$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \qquad \therefore \qquad Y = Y_1 + Y_2$$

similar to d.c. circuits

EXAMPLE

A resistance R_0 and a reactance X_0 are connected in parallel.

Determine resistance R and reactance X of the circuit.



$$G_0 = \frac{1}{R_0} \quad \text{and} \quad B_0 = -\frac{1}{X_0}$$

$$Y = G_0 + jB_0 = \frac{1}{R_0} + \frac{1}{jX_0} = \frac{1}{R_0} - \frac{j}{X_0}$$

$$Z = \frac{1}{Y} = \frac{1}{\frac{1}{R_0} - \frac{j}{X_0}} = \frac{R_0 X_0}{X_0 - jR_0} =$$

$$= \frac{R_0 X_0 \left(X_0 + jR_0\right)}{X_0^2 + R_0^2} = R_0 \frac{X_0^2}{X_0^2 + R_0^2} + jX_0 \frac{R_0^2}{X_0^2 + R_0^2}$$

$$\boxed{R = R_0 \frac{X_0^2}{X_0^2 + R_0^2} \quad \text{and} \quad X = X_0 \frac{R_0^2}{X_0^2 + R_0^2}}$$

HOW TO SOLVE PROBLEMS: *LCR* CIRCUIT

A resistor of resistance $R = 800\ \Omega$, a capacitor of capacitance $C = 1\ \mu F$ and a coil of inductance $L = 0.1$ H are connected in series to a voltage source 100 V, 200 Hz. Determine the impedance of the circuit, the current and the phase difference between the voltage and the current.

1. Draw a circuit diagram and insert all the known quantities

2. Determine the **complex** impedances of all elements. You may wish to replot the circuit to show all elements as resistors with complex impedances.

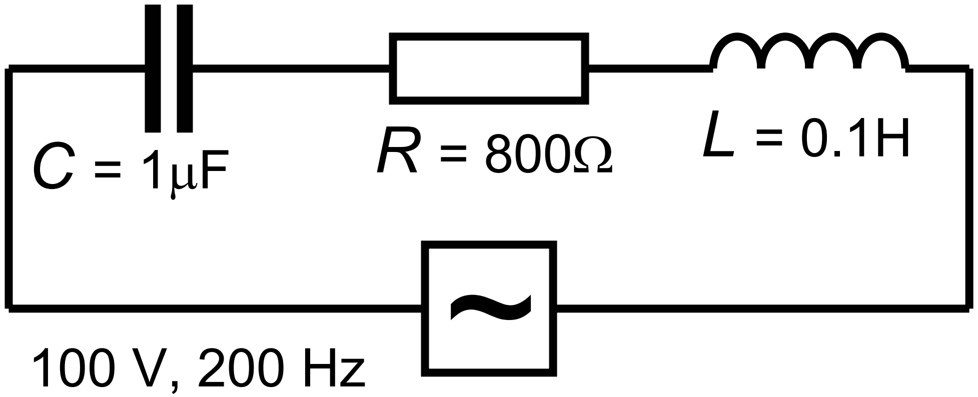
3. Determine the **complex** impedance of the circuit using the rules for d.c. circuits. Present it in the form $Z = R + X j$

4. Determine the modulus of the impedance $|Z|$ and the current $I = U / |Z|$

5. Determine the phase of the complex impedance $\varphi = \tan^{-1}(X / R)$ which is the phase shift between voltage and current. Note the sign!

PROBLEM SOLUTION

1)



2) $Z_R = 800\ \Omega$; $Z_L = (2\pi\times200\times0.1)j = 126j\ \ \Omega$;

$$Z_C = -1/(\,2\pi\times200\times10^{-6})j = -\,796j\ \ \Omega$$

3) $Z = \ 800 + (126\text{-}796)j = \ 800 - 670j\ \ [\Omega]$

4) $|Z| = \sqrt{670^2 + 800^2} = 1044\Omega$

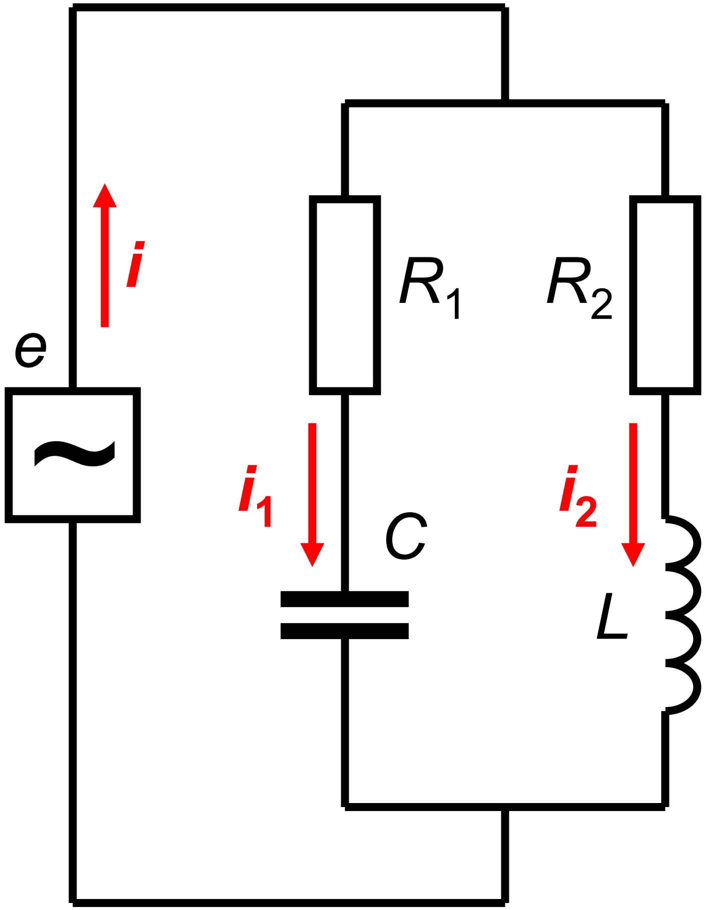
I =100/1044= 96 mA

5) $\varphi = -\tan^{-1}\left(\dfrac{796-126}{800}\right) = -40^{\text{o}}$

Voltage lags current by 40$^{\text{o}}$

Current leads voltage by 40$^{\text{o}}$

EXAMPLE: MORE COMPLICATED CIRCUIT



Known parameters: e, frequency, R_1, R_2, L, C.

Determine: current, phase angle

STEP 1

Analysis of the R_1C segment

$Z=R_1$ $\quad Z_C = -\dfrac{j}{\omega C}$

$$Z_1 = R_1 - \frac{j}{\omega C}$$

STEP 2

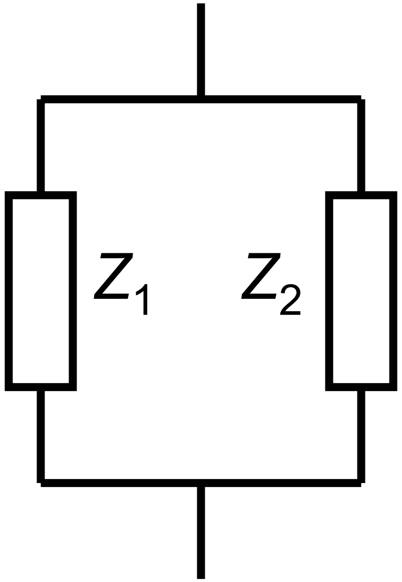
Analysis of the R_2L segment

$Z=R_1$ $\quad Z_L = j\omega L$

$$Z_2 = R_2 + j\omega L$$

STEP 3

Analysis of the whole circuit



$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

STEP 4

Current

$$I = \frac{\mathcal{E}}{|Z|}$$

Phase angle

$$Z = R + Xj$$

$$\varphi = \tan^{-1} \frac{X}{R}$$

Note

You can make calculations with complex numbers using MATLAB

However, you also need some practice using only a pocket calculator!